Problem A.3

Prove that the components of a vector with respect to a given basis are *unique*.

Solution

Let there be a vector $|\alpha\rangle$ that has components with respect to a prescribed basis.

$$|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \cdots + a_n|e_n\rangle$$

Suppose the components of this vector with respect to the basis are not unique.

$$|\alpha\rangle = b_1|e_1\rangle + b_2|e_2\rangle + \cdots + b_n|e_n\rangle$$

Some of these components can be equal; the requirement is that there's at least one value of i for which $a_i \neq b_i$. Let i_1 be one of these values. Subtract the respective sides of these equations.

$$\mathbf{0} = (a_1 - b_1)|e_1\rangle + (a_2 - b_2)|e_2\rangle + \cdots + (a_{i_1} - b_{i_1})|e_{i_1}\rangle + \cdots + (a_n - b_n)|e_n\rangle$$

Bring the term with $|e_{i_1}\rangle$ to the left side.

$$-(a_{i_1} - b_{i_1})|e_{i_1}\rangle = (a_1 - b_1)|e_1\rangle + (a_2 - b_2)|e_2\rangle + \cdots + (a_n - b_n)|e_n\rangle$$

Solve for $|e_{i_1}\rangle$.

$$|e_{i_1}\rangle = -\frac{1}{a_{i_1} - b_{i_1}}\left[(a_1 - b_1)|e_1\rangle + (a_2 - b_2)|e_2\rangle + \cdots + (a_n - b_n)|e_n\rangle\right]$$

Since one of the basis vectors can be expressed in terms of the others, the vectors are linearly dependent. This is a contradiction because the vectors that form a basis are linearly independent by definition. Therefore, the components of a vector with respect to a given basis are unique.